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Finite-size scaling studies of the O(2) Heisenberg spin model in $(D+1)$ dimensions

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Abstract. The finite-size scaling behaviour of the O(2) Heisenberg spin model in $(D+1)$ dimensions, $D = 1$ and 2, is computed in the 'low-temperature' region using the stochastic truncation Monte Carlo method. The numerical results so obtained agree very well with effective Lagrangian theory, and with spin-wave perturbation theory.

1. Introduction

There have recently been important advances in the theory of finite-size scaling, for systems with a continuous global symmetry which undergo a first-order phase transition, or spontaneous symmetry breaking. According to Goldstone's theorem (1961), such systems are expected to exhibit Goldstone bosons. The massless Goldstone bosons then control the behaviour of the system at low energies or large distances. One may write down a continuum 'effective Lagrangian' for the Goldstone bosons, specified purely in accordance with the symmetry properties of the model. For a lattice spin model, *universal* predictions can then be given for the leading finite-size corrections, low-temperature corrections, and small magnetic field corrections to the bulk behaviour, in terms of just two or three parameters: e.g. the spin-wave stiffness, the spin-wave velocity and the spontaneous magnetization. Much of this was already recognized in earlier work by Cardy and Nightingale (1983), Fisher and Privman (1985), Neuberger and Ziman (1989), Fisher (1989) and others. Recently, Gasser and Leutwyler (1988), Hasenfratz and Leutwyler (1990) and Hasenfratz and Niedermayer (1992) have shown that one may develop a systematic expansion at low energies or large volume, based on the fact that the interaction between the Goldstone bosons is 'soft', that is, weak at low energies. Hence higher-order correction terms can be derived.

The effective Lagrangian approach thus provides a systematic and universal field theoretic description of the behaviour at a *first-order* transition, for systems with a continuous symmetry, on a footing similar to that for second-order transitions. The unknown parameters are the renormalized couplings of the Goldstone bosons, rather than a set of critical indices. The predictions are generally valid for dimensions $D > 2$, where spontaneous symmetry breaking occurs; but as we shall see, they remain at least partially valid even for $D = 2$.

In the present work, we set out to test these predictions, via numerical calculations of the finite-size scaling behaviour of the O(2) Heisenberg spin model in $(D+1)$ dimensions. The

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method used is the Monte Carlo technique of symmetrized stochastic truncation, discussed recently by Price *et al* (1993) in connection with the Ising model. The algorithm is slow, but very accurate, and gives reliable results for lattices of small to moderate size.

First of all, the predictions of effective Lagrangian theory for this 'cylindrical' geometry, which were derived by Hasenfratz and Niedermayer (1993), are summarized. Then the complementary predictions of spin-wave theory, which were derived by Hamer and Zheng (1992), are also summarized. Spin-wave theory leads to series expansions for the finite-size scaling correction terms, rather than exact expressions; but on the other hand, it does give predictions for the low-energy parameters such as the spin-wave velocity, which effective Lagrangian theory cannot. The two approaches are completely consistent with each other, up to the order of spin-wave theory calculated so far.

Comparing the numerical results in the 'low-temperature' region of this model with the theoretical predictions, the agreement is found to be excellent. Even the smallest lattices seems to follow the behaviour predicted by the theory. For the $(1+1)$-dimensional model, there is no spontaneous symmetry breaking, and neither theory is strictly applicable; but they happen to agree with conformal theory for the non-magnetic observables in this case, and turn out to work rather well. There is only one independent parameter at leading order in this model, which may be taken as the critical index η. A combination of spin-wave theory with the theory of Kosterlitz and Thouless (1973, 1974) accounts for the behaviour of η quite well.

For the $(2+1)$-dimensional model, there turn out to be two independent low-temperature parameters of interest, the spin-wave velocity v and spontaneous magnetization Σ. Estimates of these quantities are obtained, which again agree quite well with spin-wave predictions. The behaviour near the second-order phase transition at the end-point of the first-order line is also explored. Estimates of the critical indices are obtained, but they do not yet compare in accuracy with Euclidean Monte Carlo or series analysis results.

We conclude that effective Lagrangian theory and spin-wave perturbation theory together provide a very good description of the finite-size scaling behaviour of these systems in the 'low-temperature' region.

2. Theoretical results

2.1. *Effective Lagrangian theory*

Hasenfratz and Niedermayer (1993) have derived the predictions of effective Lagrangian theory in the 'cylindrical' geometry appropriate to a Hamiltonian model in $(D+1)$ dimensions. They specifically discussed the quantum Heisenberg antiferromagnet, but the results are also applicable to the $\mathrm{O}(N)$ ferromagnet. The effective Lagrangian for the Goldstone bosons (magnons) involves three low-energy parameters, the spin-wave stiffness or helicity modulus ρ_S, the spin-wave velocity v, and the spontaneous magnetization Σ. Thus Hasenfratz and Niedermayer (1993) predict that at zero temperature, on a lattice of L^D sites with periodic boundary conditions, the following finite-size scaling behaviours will hold:

(i) Finite-size corrections to the ground-state energy density, $\epsilon_0(L) = E_0/L^D$

$$\epsilon_0(L) - \epsilon_0(\infty) = -\frac{(N-1)v}{2L^{D+1}}\left[\alpha^{(D)}_{-1/2}(1) + 2 - \frac{2}{D+1}\right] + \mathrm{O}(L^{-2D}) . \quad (2.1)$$

(ii) Finite-size spectrum of low-energy excitations, in the zero mode sector

$$E_j - E_0 = \frac{j(j+N-2)v^2}{2\rho_S L^D} + O\left(L^{1-2D}\right). \tag{2.2}$$

(iii) Finite-lattice longitudinal susceptibility

$$\chi_\parallel = \frac{4\Sigma^2 \rho_S L^{2D}}{N(N-1)v^2} + O\left(L^{D+1}\right). \tag{2.3}$$

In equation (2.1), $\alpha^{(D)}_{-1/2}(1)$ is a numerical structure factor tabulated by Hasenfratz and Leutwyler (1990).

2.2. Spin-wave theory

Spin-wave expansions for the O (2) Heisenberg spin model in $(D+1)$ dimensions have been discussed by Hamer and Zheng (1992, hereafter referred to as I). The quantum Hamiltonian for the model can be written as (Hamer *et al* 1979)

$$H = \sum_i J^2(i) - x \sum_{\langle ij \rangle} \cos[\theta(i) - \theta(j)] - h \sum_i \cos\theta(i) \tag{2.4}$$

where $\langle i, j\rangle$ denotes nearest-neighbour pairs, $\theta(i)$ is an angle variable at site i, and $J(i)$ is the angular momentum operator conjugate to $\theta(i)$:

$$\left[J(i), \theta(j)\right] = -i\delta_{ij}. \tag{2.5}$$

The parameter x is related to the inverse temperature in the Euclidean formulation, and h is the magnetic field.

By Taylor expanding the cosine, Fourier transforming, and making a Bogoliubov transformation, one obtains an equivalent boson Hamiltonian with interactions which can be treated by Rayleigh–Schrödinger perturbation theory, giving rise to an expansion for the thermodynamic functions in powers of $(xz)^{-1/2}$, where z is the co-ordination number of the lattice.

The spin-wave expansion so obtained agrees precisely, to the order calculated, with the finite-size scaling behaviour predicted by effective Lagrangian theory, and in addition it provides explicit series expansions for the parameters ρ_S, v, and Σ. The results relevant here are as follows:

(i) Spin-wave velocity:
one-dimensional chain

$$v = \sqrt{2x} - (1/\pi) - 0.06126111x^{-1/2} + O\left(x^{-1}\right) \tag{2.6a}$$

two-dimensional triangular lattice

$$v = \sqrt{3x} - 0.241936 - 0.0214988x^{-1/2} + O\left(x^{-1}\right). \tag{2.6b}$$

(ii) Spontaneous magnetization:
one-dimensional chain

$$\Sigma = 0 \tag{2.7a}$$

two-dimensional triangular lattice

$$\Sigma = 1 - 0.178708x^{-1/2} - 0.008994x^{-1} - 0.001107x^{-3/2} + \mathrm{O}(x^{-2})\,. \tag{2.7b}$$

(The spin-wave expansion for the one-dimensional chain does not converge; but the Mermin–Wagner theorem (1966) assures us that the spontaneous magnetization is strictly zero for any finite x, in this case.)

The finite-lattice mass gap is found to be

$$E_1 - E_0 = \frac{1}{L^D} \tag{2.8}$$

exact to all orders in the perturbative spin-wave expansion, with the only corrections being due to non-perturbative effects. Comparing equation (2.8) with (2.2) it follows that an identity holds for this model†

$$\frac{v^2}{2\rho_S} = 1 \tag{2.9}$$

The spin-wave stiffness was not calculated directly in I, but a brief calculation shows that (2.9) is correct, to the first two orders in $x^{-1/2}$.

The fact that the effective Lagrangian theory gives an exact relationship between spin-wave velocity and the leading finite-size correction to the ground-state energy, which should be true to all orders in the spin-wave expansion, implies an interesting *diagrammatic identity* in the spin-wave theory; but we shall not pursue that point here.

The spin-wave expansion is in fact closely related to the large-volume expansion in the effective Lagrangian theory, as discussed by Hasenfratz and Niedermayer (1993). The two approaches differ in that the continuum effective theory makes universal predictions in terms of unknown parameters v, ρ_S, and Σ, whereas the spin-wave expansion gives explicit expansions for these parameters, for each particular model.

For the $(1+1)$-dimensional model, neither the effective Lagrangian theory nor the spin-wave theory are strictly applicable, since there is no spontaneous symmetry breaking at finite x. Nevertheless, both give sensible predictions for the non-magnetic observables, which agree with each other, provided that equation (2.9) is satisfied, again, to all orders in the spin-wave expansion. They also agree with conformal theory for a critical model in $(1+1)$ dimensions (Cardy 1987), which predicts

$$\epsilon_0(L) - \epsilon_0(\infty) \sim -\frac{\pi v c}{6L^2} \tag{2.10}$$

and

$$E_1 - E_0 \sim \frac{\pi v \eta}{L} \tag{2.11}$$

provided the conformal anomaly $c = 1$ (as is well known for the O(2) model in two dimensions), and the critical index η is related to the spin-wave velocity v by

$$\pi v \eta = 1\,. \tag{2.12}$$

† We take this opportunity to correct a serious error in I, where an incorrect formula for the mass gap was given, resulting in an incorrect relation between the spin-wave velocity and the spin-wave stiffness.

3. Method and results

To carry out the Monte Carlo calculations, we used the method of 'symmetrized' stochastic truncation, which has recently been discussed by Price *et al* (1993). We shall not repeat details here. It is based on the power method, in which the Hamiltonian is multiplied repeatedly into a trial vector, until the dominant eigenvector is *projected out*. Price *et al* (1993) treated the Ising model; the differences which arise in connection with the O (2) model are as follows:

(i) Strong coupling basis. In a strong coupling or angular momentum representation, the Hamiltonian (2.4) can be rewritten as

$$H = \sum_i J^2(i) - \frac{x}{2}\sum_{\langle ij\rangle}\left[J_+(i)J_-(j) + J_-(i)J_+(j)\right] - h\sum_i\left[J_+(i) + J_-(i)\right] \tag{3.1}$$

where $J_{\pm}(i)$ are raising/lowering operators for the spin J at site i:

$$\left[J(i), J_{\pm}(j)\right] = \pm J_{\pm}(i)\delta_{ij}\,. \tag{3.2}$$

The basis states are chosen as eigenstates of the spin $J(i)$ at each site, which can take any integer value. There are thus an infinite number of spin states at any site, whereas in the Ising model there are only two.

(ii) Spin cutoff. To apply the stochastic truncation method, one must arrange that the ground-state is the *dominant* eigenstate (with the largest eigenvalue magnitude), which can be simply achieved by working with the modified Hamiltonian

$$H' = E_{\text{cut}} - H \tag{3.3}$$

where E_{cut} is a suitable large energy cutoff. To make sure that the system does not converge on some unwanted very *high* spin state instead, we apply a cutoff

$$J(i) \leqslant J_{\text{cut}} \tag{3.4}$$

$$E_{\text{cut}} = L^D J_{\text{cut}}^2\,. \tag{3.5}$$

We have checked that J_{cut} is chosen large enough to have no effect on the present calculations, within errors ($J_{\text{cut}} \geqslant 7$ for most of the calculations in this paper).

(iii) Variational guidance. The accuracy of the method is strongly enhanced if a good approximation to the ground-state wavefunction can be found to provide *variational guidance* (DeGrand and Potvin 1985, Price *et al* 1993). We used a simple one-parameter variational wavefunction:

$$|\chi_0\rangle = \exp\left(-c\sum_i J^2(i)\right) \tag{3.6}$$

for this purpose, where the parameter c has to be optimized at each separate coupling. There has been no sign in previous applications that this procedure introduces any systematic bias into the results.

(iv) Efficiency. The 'symmetrization' procedure is very expensive in computer time, and the symmetrized stochastic truncation method is only efficient if the average 'occupation number' $\langle n\rangle$ of the basis states in the ensemble is high (Price *et al* 1993). For most of the calculations reported in this paper $\langle n\rangle$ was of order 10^3, but for large lattice sizes and large couplings it may become very much lower than this.

At each lattice size and each coupling, test runs were performed to check that the energy estimates had reached equilibrium, and were independent of ensemble size, and to optimize the variational parameter c. No details of these tests will be shown.

3.1. (1 + 1)-dimensional O(2) model, low-temperature region

For the $D = 1$ case, Monte Carlo data have been obtained for chains of up to 10 sites out to couplings $x = 8$, using periodic boundary conditions.

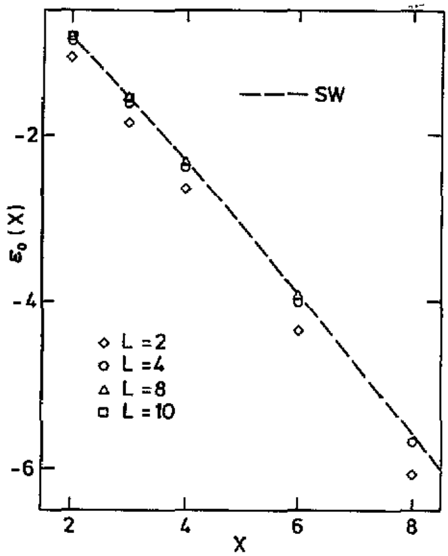

Figure 1. The ground-state energy density ϵ_0 as a function of x, for the $(1+1)$-dimensional model, lattice sizes $L = 2, 4, 8, 10$. The line is the spin-wave prediction for the bulk limit.

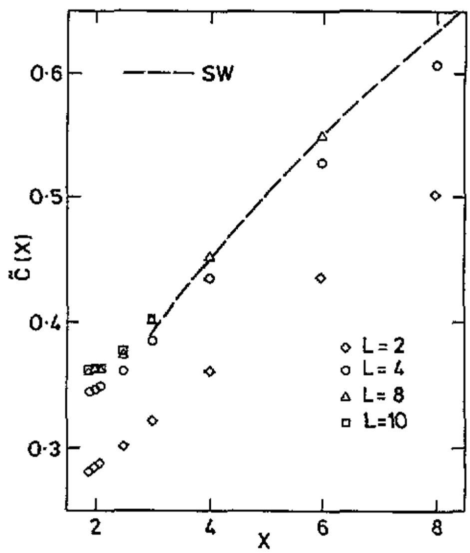

Figure 2. The 'specific heat' $\tilde{C}(x)$ as a function of x for the $(1+1)$-dimensional model. Notation as in figure 1.

Figure 1 shows the ground-state energy per site as a function of x. The spin-wave prediction for this quantity is

$$\epsilon_0(\infty) = -x + \frac{2\sqrt{2x}}{\pi} - \frac{1}{\pi^2} - 0.01080x^{-1/2} + \mathrm{O}\left(x^{-1}\right). \tag{3.7}$$

It can be seen that the finite-lattice values converge towards the spin-wave prediction in the bulk limit, over pratically the entire range of couplings $x > 2$.

Figure 2 shows the 'specific heat'

$$\tilde{C}(x) = -\frac{x^2}{L}\frac{\partial^2 E_0}{\partial x^2} \tag{3.8}$$

as a function of x. The spin-wave prediction describes the data quite accurately for $x \geqslant 4$. At around $x = 2$ (the critical point), the data develop a shoulder, but no divergent peak; this is the expected behaviour for a Kosterlitz–Thouless transition.

We turn next to the finite-size behaviour. Figure 3 shows the ground-state energy density as a function of $1/L^2$, for various couplings x. The data are well fitted by straight lines, as

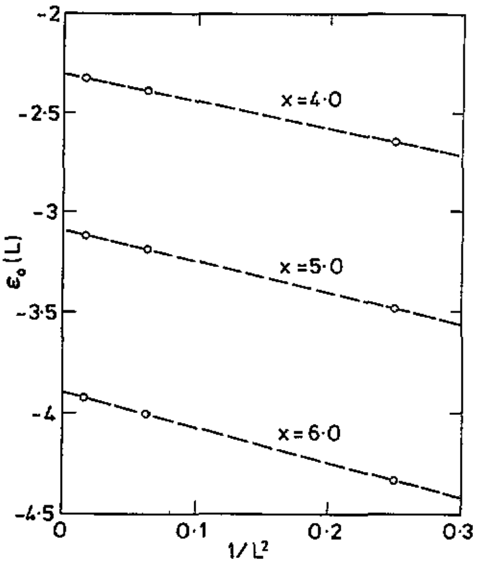

Figure 3. The ground-state energy density ϵ_0 as a function of $1/L^2$, where L is the lattice size, for the (1+1)-dimensional model. Straight line fits are shown.

predicted by the formula

$$\epsilon_0(L) \sim \epsilon_0(\infty) - \frac{1}{6\eta L^2} \qquad \text{as } L \to \infty \tag{3.9}$$

which follows from (2.1) and (2.12), and hence one can extract estimates for $\epsilon_0(\infty)$ and η. These estimates are listed in table 1 along with the spin-wave predictions. The agreement is very good for the ground-state energy. The data for η are about 6% lower than the spin-wave predictions.

Table 1. Comparison of spin-wave predictions with values estimated from Monte Carlo data for the (1 + 1)-dimensional chain. Quantities listed are the bulk ground-state energy density $\epsilon_0(\infty)$ and the critical index η.

		Coupling, x			
		3.0	4.0	5.0	6.0
$\epsilon_0(\infty)$	Data	−1.5479(3)	−2.304(1)	−3.088(2)	−3.897(3)
	Prediction	−1.5482(3)	−2.3061(3)	−3.0930(3)	−3.9004(3)
η	Data	0.1418(5)	0.120(7)	0.106(7)	0.096(5)
	Prediction	0.1519(3)	0.1284(3)	0.1130(3)	0.1020(2)

Figure 4 shows the mass gaps as a function of x, for different lattice sizes L. They level out at a constant value

$$F_L(x) = \frac{1}{L} \tag{3.10}$$

for $x \geqslant 4$, in excellent agreement with the spin-wave prediction. This behaviour was already remarked by Roomany and Wyld (1980) and Hamer and Barber (1980, 1981).

Figure 5 shows a logarithmic plot of the parallel susceptibility $\chi(L)$ as a function of lattice size L, at coupling $x = 5$. The data are well fitted by a straight line, in agreement

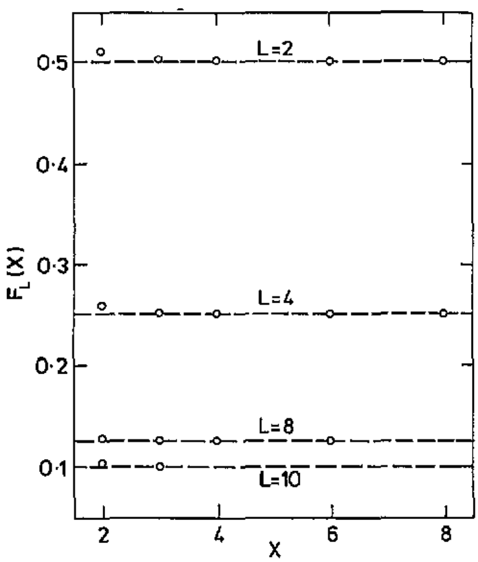

Figure 4. The mass gap $F_L(x)$ as a function of x for the $(1+1)$-dimensional model. The straight lines are the spin-wave predictions.

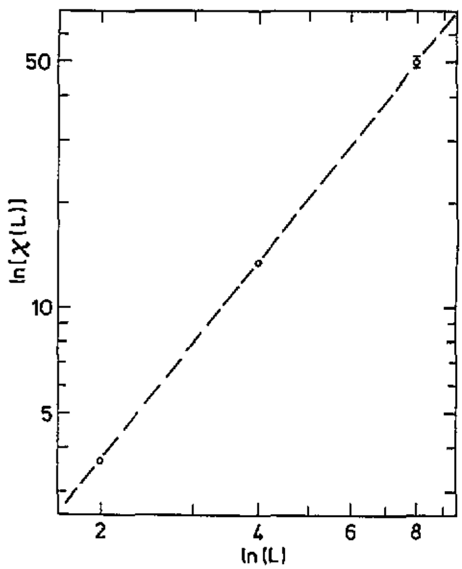

Figure 5. A logarithmic plot of the longitudinal susceptibility $\chi_{\parallel}$ as a function of lattice size L, for the $(1+1)$-dimensional model. A straight line fit is shown.

with the behaviour expected from finite-size scaling theory

$$\chi(L) \sim L^{2-\eta} \tag{3.11}$$

and hence one can obtain another estimate of η. The result lies about 5% above the spin-wave prediction in this case.

Now Kosterlitz and Thouless (1973) and Kosterlitz (1974) have shown that in the vicinity of the critical point, η approaches a universal value of $\frac{1}{4}$:

$$\eta \sim \tfrac{1}{4}c - \alpha(x - x_c)^{1/2} \qquad \text{as } x \to x_c\,. \tag{3.12}$$

Following Luck's (1982) analysis for the Euclidean case, we have used equations (2.6a) and (2.12) to calculate a spin-wave expansion for the quantity $(\frac{1}{4} - \eta)^2$ as a function of the variable $x^{-1/2}$:

$$\left(\tfrac{1}{4} - \eta\right)^2 = \tfrac{1}{16} - 0.11254x^{-1/2} + 0.025330x^{-1} + 0.012229x^{-3/2} + \mathrm{O}(x^{-2})\,. \tag{3.13}$$

This function should pass linearly through zero at the critical point. Through the order given in equation (3.13), the zero occurs at

$$x_c \simeq 2.01 \tag{3.14}$$

which agrees very well with the values obtained by numerical methods (Beleznay 1986, Allton and Hamer 1988), although it is hard to fix the position of a Kosterlitz–Thouless transition to better than a few percent. Figure 6 shows a plot of the numerical values obtained here against the prediction (3.13). It can be seen that the numerical results lie somewhat above the spin-wave prediction in the vicinity of the critical point. This may very well be due to logarithmic corrections to scaling at the critical point (Allton and Hamer 1988), which are known to occur in the similar case of the XXZ Heisenberg model (Alcaraz *et al* 1987). They result in extremely slow convergence of the finite-lattice results.

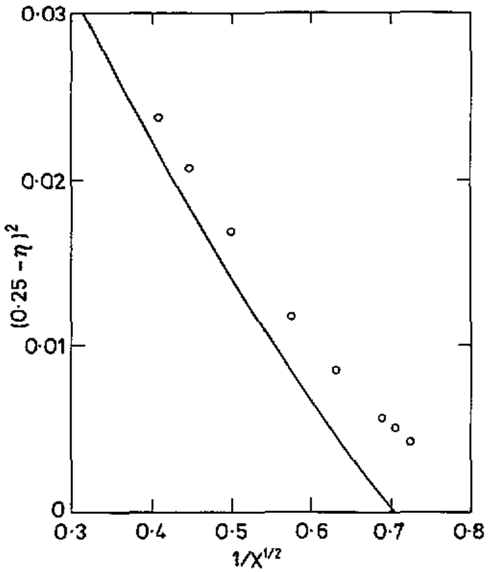

Figure 6. A plot of $(\frac{1}{4} - \eta)^2$ against $1/\sqrt{x}$, where η is the critical index, for the (1 + 1)-dimensional model. The solid line is the spin-wave prediction.

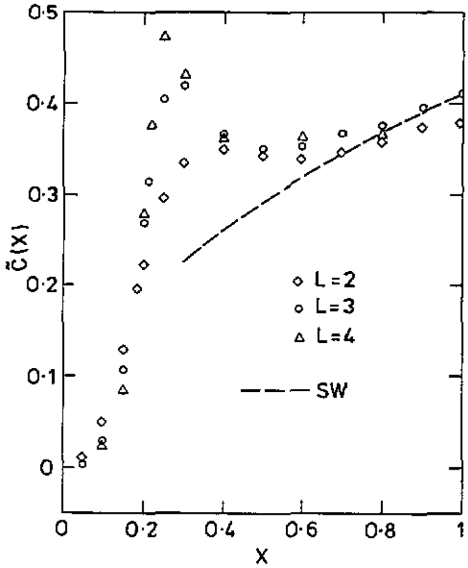

Figure 7. The 'specific heat' $\tilde{C}(x)$ as a function of x for the (2+1)-dimensional model, lattice sizes $L = 2, 3, 4$. The line is the spin-wave prediction for the bulk limit.

3.2. (2 + 1)-dimensional O(2) model, low-temperature region

For the $D = 2$ case, numerical results have been calculated for triangular lattices up to 4×4 sites out to couplings $x = 1$, using periodic boundary conditions. Some results were also obtained for the 5×5 lattice, but their accuracy was generally not sufficient to justify presenting them here.

The bulk ground-state energy per site obeys the spin-wave prediction

$$\epsilon_0(\infty) = -3x + 1.67618\sqrt{x} - 0.117066 - 0.00669x^{-1/2} + O(x^{-1}) \quad (3.15)$$

very well for $x \geqslant 0.4$. The *specific heat* defined by equation (3.9) is plotted against x in figure 7. At large x the results appear to converge towards the spin-wave theory, but around $x \simeq 0.2$ a small but increasing sequence of finite lattice peaks emerges, indicating the expected second-order critical point.

The finite-size behaviour of the ground-state energy density is illustrated in figure 8. The data agree well with the predicted form of

$$\epsilon_0(L) \sim \epsilon_0(\infty) - \frac{0.7802v}{L^3} \qquad \text{as } L \to \infty \quad (3.16)$$

(where 0.7802 is the structure factor coefficient appropriate to the triangular lattice—see the appendix of I) and hence values for $\epsilon_0(\infty)$ and v can be extracted, which are listed in table 2, along with the spin-wave predictions. Once again, it can be seen that the spin-wave predictions describe the data very well for $x \geqslant 0.4$. There is no sign of any dramatic change in behaviour as the critical point is approached, and equation (3.16) may well still hold at the second-order critical point.

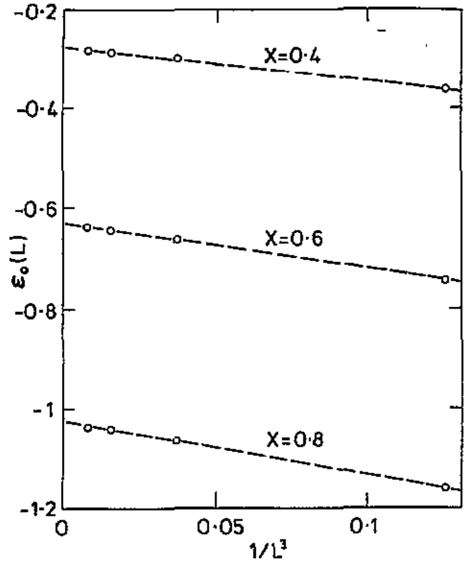

Figure 8. The ground-state energy density ϵ_0 as a function of $1/L^3$, where L is the lattice size, for the (2+1)-dimensional model. Straight line fits are shown.

Table 2. Comparison of spin-wave predictions with values estimated from Monte Carlo data, for the (2 + 1)-dimensional triangular lattice. Quantities listed are the bulk ground-state energy density $\epsilon_0(\infty)$ and the spin-wave velocity v.

		Coupling, x		
		0.4	0.6	0.8
$\epsilon_0(\infty)$	Data	−0.274(1)	−0.628(1)	−1.025(2)
	Prediction	−0.268(2)	−0.627(2)	−1.025(1)
v	Data	0.925(7)	1.1(1)	1.36(7)
	Prediction	0.82(1)	1.072(7)	1.282(6)

Figure 9 shows the finite-lattice mass gap as a function of x. As in the (1 + 1)-dimensional case, it drops steeply and then flattens out at the constant value

$$F_L(x) = \frac{1}{L^2} \tag{3.17}$$

for larger x, in excellent agreement with the spin-wave prediction.

The parallel susceptibility $\chi(L)$ also appears to agree with the theoretical prediction that it should increase like L^4, at large coupling x, although it becomes somewhat unstable and slow to calculate in this region.

3.3. (2 + 1)-dimensional O(2) model, critical region

We have attempted to carry out a finite-size scaling analysis of the data in the critical region, $x \simeq 0.2$, in order to estimate the critical parameters. We follow the methods of Price *et al* (1993). The pseudo-critical point x_L at lattice size L is defined in the usual way as the coupling such that the scaled mass-gap ratio passes through unity

$$R_L(x) = 1 \tag{3.18}$$

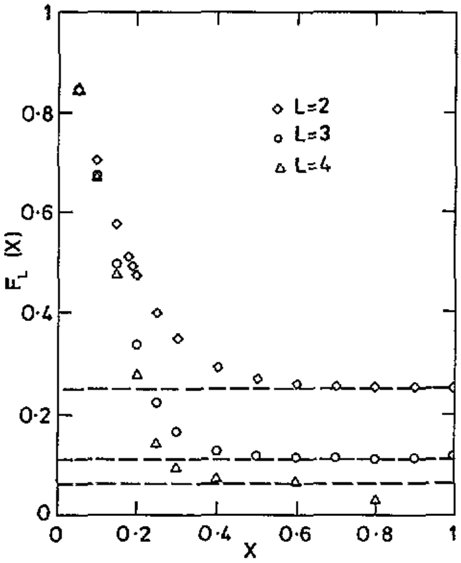

Figure 9. The mass gap $F_L(x)$ as a function of x for the (2 + 1)-dimensional model. The straight lines are the spin-wave predictions.

where

$$R_L(x) = \frac{LF_L(x)}{(L-1)F_{L-1}(x)} \tag{3.19}$$

$F_L(x)$ being the mass gap for lattice size L. The pseudo-critical points were found by an iterative search method for each lattice size L, $L = 1, \ldots, 4$. Next, the thermodynamic quantities of interest were estimated for both the lattice size L and $L - 1$ corresponding to each pseudo-critical point x_L. The results are displayed in table 3.

Table 3. Finite-lattice data for the (2 + 1)-dimensional triangular lattice at the pseudo-critical points x_L. The ground-state energy density and its first two derivatives with respect to x, the mass gap and its first two derivatives, and the susceptibility are given.

x_L	$x_2 = 0.185\,628(5)$		$x_3 = 0.213\,078(7)$		$x_4 = 0.218\,038(9)$	
L	1	2	2	3	3	4
ϵ_0	−0.278 442	−0.072 7431(6)	−0.098 991(2)	−0.060 1899(6)	−0.064 1301(7)	−0.044 237(3)
ϵ_0'	−1.5	−0.878 90(2)	−1.032 04(5)	−0.777 34(2)	−0.811 74(2)	−0.726 75(6)
ϵ_0''	0.0	−5.706 5(4)	−5.412 (1)	−6.923 7(6)	−6.951 1(5)	−7.917 (3)
F	1.0	0.500 000(4)	0.451 259(9)	0.300 825(7)	0.288 712(7)	0.216 67(6)
F'	0.0	−1.933 4(1)	−1.620 2(2)	−2.499 5(2)	−2.383 0(2)	−2.983(1)
F''	0.0	11.492 (2)	11.164 (5)	23.271 (6)	23.746 (5)	39.34(5)
χ	2.0	3.728 24(2)	4.457 39(2)	9.808 77(5)	10.530 53(4)	18.483 5(9)

The sequence of pseudo-critical points converges very rapidly, as in the Ising case, and a plot against $1/L^4$ (figure 10) gives an extrapolation to the bulk limit

$$x_c = 0.2203(3) \tag{3.20}$$

which is in reasonable agreement with the result $x_c = 0.2207(1)$ obtained from a series analysis by Hamer and Guttmann (1989).

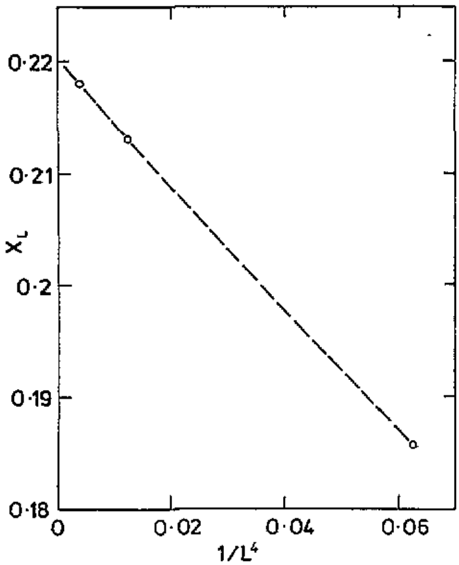

Figure 10. The pseudo-critical coupling x_L as a function of $1/L^4$, where L is the lattice size, for the (2 + 1)-dimensional model. The dashed line is to guide the eye. The straight line is the spin-wave prediction.

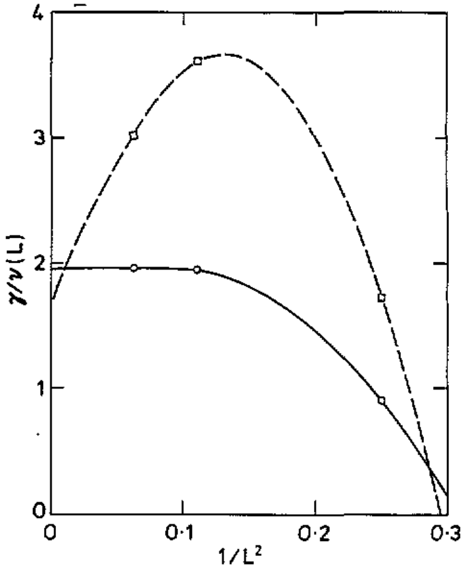

Figure 11. Estimates of γ/ν plotted against $1/L^2$, for the (2 + 1)-dimensional model. The solid line connects the 'logarithmic' estimates, while the dashed one connects the 'linear' estimates.

or a 'logarithmic' estimate

$$\frac{\ln[\chi_L(x_L)/\chi_{L-1}(x_{L-1})]}{\ln[L/L+1]} \sim \frac{\gamma}{\nu} \qquad \text{as } L \to \infty \tag{3.22}$$

where $\chi_L(x)$ is the finite-lattice susceptibility.

Similar estimates for the ratios $1/\nu$, α/ν can be obtained from the finite-lattice beta function and specific heat, respectively. As an example, the results for γ/ν are plotted against $1/L^2$ in figure 11. Since there are only three points on each line, one can only perform a crude extrapolation to the limit $L \to \infty$, obtaining final estimates of

$$1/\nu = 1.47(5) \qquad \text{or } \nu = 0.68(3) \tag{3.23}$$

$$\gamma/\nu = 1.96(5) \qquad \text{or } \gamma = 1.33(5) \tag{3.24}$$

$$\alpha/\nu = 0.2(1) \qquad \text{or } \alpha = 0.14(7)\,. \tag{3.25}$$

These may be compared with the series estimates of Hamer and Guttmann (1989):

$$\nu = 0.686(3) \qquad \gamma = 1.334(5)\,. \tag{3.26}$$

It can be seen that our present results are in good agreement with the series estimates, but are much less accurate. They are also much less accurate than current Euclidean Monte Carlo estimates for this model, such as those of Janke (1990)

$$\nu = 0.670(2) \qquad \gamma = 1.316(5) \tag{3.27}$$

obtained from lattices of up to 48^3 sites. However, our results do represent the first application of a Monte Carlo method to the Hamiltonian version of this model, as far as we are aware. If useful results could be obtained for the 5×5 lattice, the exponent estimates could be greatly improved.

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4. Summary and conclusions

The method of stochastic truncation has been used to perform numerical Monte Carlo calculations of the O (2) Heisenberg spin model in (1 + 1) and (2 + 1) dimensions. Our main object was to explore the finite-size scaling behaviour of these models in the 'low-temperature' or large-x region, and to compare the results with the predictions of effective Lagrangian theory (Hasenfratz and Niedermayer 1993) and spin-wave perturbation theory (Hamer and Zheng 1992). The general conclusion is that the numerical results agree very well with the theoretical predictions.

For the (1+1)-dimensional model, there is no spontaneous symmetry breaking (Mermin and Wagner 1966), and neither of the two theoretical approaches is strictly applicable; but it is found that their predictions works rather well for the non-magnetic observables, and agree with conformal theory (Cardy 1987) where they overlap. The critical index η is related to the spin-wave velocity v in this model by

$$\pi v \eta = 1 \tag{4.1}$$

which leaves only one parameter (η, say) to control the 'low-temperature' behaviour. The finite-size correction to the ground-state energy is

$$\epsilon_0(L) - \epsilon_0(\infty) \sim -\frac{1}{6\eta L^2} \qquad \text{as } L \to \infty \tag{4.2}$$

and the mass gap is

$$E_1 - E_0 = \frac{1}{L} \tag{4.3}$$

exact to all orders in the spin-wave expansion. The finite-lattice susceptibility is

$$\chi_L \sim L^{2-\eta}\,. \tag{4.4}$$

Equations (4.2) and (4.4) have been used to obtain estimates of η as a function of coupling x. These estimates are presented in table 1 and figure 6; they agree well with the predictions of spin-wave theory.

For the (2 + 1)-dimensional model, we have found a different relationship between the parameters of the effective Lagrangian:

$$\frac{v^2}{2\rho_S} = 1 \tag{4.5}$$

where ρ_S is the spin-wave stiffness. Then the finite-size correction to the ground-state energy is

$$\epsilon_0(L) - \epsilon_0(\infty) \sim -\frac{0.7189v}{L^3} \qquad \text{as } L \to \infty \tag{4.6}$$

and the mass gap is

$$E_1 - E_0 = \frac{1}{L^2} \tag{4.7}$$

exact to all orders in spin-wave theory; while the susceptibility obeys

$$\chi_L \sim \Sigma^2 L^4 \qquad \text{as } L \to \infty \tag{4.8}$$

where Σ is the spontaneous magnetization. The resulting estimates of v are displayed in table 2, and again agree quite well with the spin-wave predictions.

We have also obtained some estimates of the critical parameters at the second-order transition for the $(2+1)$-dimensional model, using the same methods as Price *et al* (1993). The results were somewhat crude, but were in rough agreement with the series estimates of Hamer and Guttmann (1989).

Euclidean Monte Carlo simulations of the O (2) model have been carried out recently on lattices of enormous size: e.g. $L = 512$ by Gupta *et al* (1988) and $L = 1200$ by Janke and Nather (1991) for the 2D model, while for the 3D model Hasenbusch and Meyer (1990) went up to $L = 16$, while Janke (1990) reached $L = 48$. Biferale and Petronzio (1989) performed an analysis of η in the low-temperature region of the 2D model, and obtained a behaviour much the same as that seen here.

One could hardly contemplate treating lattices of such size with the stochastic truncation method. But our results confirm the conclusion of Price *et al* (1993), that stochastic truncation is a reliable and accurate method for Hamiltonian lattices of small to moderate size; and as is by now well known, a surprising amount can be learned by careful analysis of the finite-size scaling behaviour of such systems.

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